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0003-682X(95)00033-X

Applied Acoustics 47 (1996) 67–81
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0003-682X/96/$15.00

A Statistical Approach for the Diffusion of Sound by Reflecting Surfaces in Large Enclosures

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(Received 7 October 1994; revised version received 17 May 1995; accepted 1 June 1995)

ABSTRACT

This paper is concerned with the applicability of a time series model for the separation of the reflected noise energy by rough reflecting surfaces into diffuse and geometrically reflected noise. For the derivation of the theoretical model a combination of theory of geometrical acoustics and statistical acoustics was used. Here the measured noise at the receiver position is assumed to be the superposition of three types of noise: the direct which follows the inverse square law, the geometrically reflected which follows Snell's law, and the diffuse noise which propagates omnidirectionally and decays following an exponential law. The solution of the problem is finally achieved in the time domain statistically, by the use of time series analysis.

Keywords: Diffusion, reverberation, time series.

1 INTRODUCTION

For an accurate determination of the reverberation time of a room, the basic assumption is that the noise field is completely diffuse. A diffuse sound field is such that at any position, sound waves are incident from all directions with equal intensities and random phase relations. Since the degree of diffusion is related to the accuracy of measurement, it must be properly defined. In most cases the commonly used models are based on a combination of the direct and the reverberant field. The reverberant field, which sometimes is regarded as diffuse, is the field that is made up of waves which have been reflected at least once from various surfaces of the room. In some cases, as for example with low absorption the reflected noise energy and hence the corresponding reverberant field have properties closer to the direct than the diffuse noise field.

In the present paper we assume the reverberant field is diffuse and try to find statistically a relation between the factors of diffusion and geometrically reflected noise. According to this method the idealised sound field at the observation point at any time consists of the superposition of three components: the direct noise, with duration close to one mean free path time, the geometrically reflected noise, with duration several mean free path times, and the diffuse noise field, which has the longer life, close to the reverberation time T_{60}. The reverberant sound is assumed to propagate omnidirectionally and to decay exponentially. The reflected noise following the laws of geometrical acoustics is regarded as direct sound emitted by image sources lying in virtual space. The evaluation of the degree of diffusion by each reflecting surface is achieved using time series analysis and is based on the time arrival of energy to the observation point. Since the model is a statistical one, it deals with the mean values and the deviations about the mean, neglecting the nature of sound waves and the characteristics of the reflecting surface. This is the basic advantage of the method.

2 MODEL ESTABLISHMENT

As a descriptor of noise we use the noise energy, which is a linear quantity and which can be described by the average relative squared noise pressures. As a source of noise, a white noise source is used. The latter, in the statistical sense, is a source emitting a series of randomly distributed impulses equally spaced in the time axis.

Since the time series used here does not have zero mean, we subtract the mean m from each measurement to obtain a zero mean series. In this paper we use y to denote the original data (noise energy), and z for the deviations from the mean. We exclude from consideration regions too close to the source and too close to the boundaries of the room. The dimension of the room under consideration is large compared with the greatest wavelength of the frequency band under consideration. The reflecting ability of each surface depends upon the angle of incidence. The calculated geometrically reflected noise field at the observation point is a sum of reflected rays corresponding to random incidence and the values of the parameters used are the mean values. In addition we assume that the reverberant field at the received position is representative of that in the room as a whole.

Under the assumptions stated above, in this model, we divide the reflected energy fraction $(1\text{-}a)$ into two terms, namely the term $g(1\text{-}a)$ which is the fraction of speculary reflected energy and the term $(1\text{-}g)(1\text{-}a)$

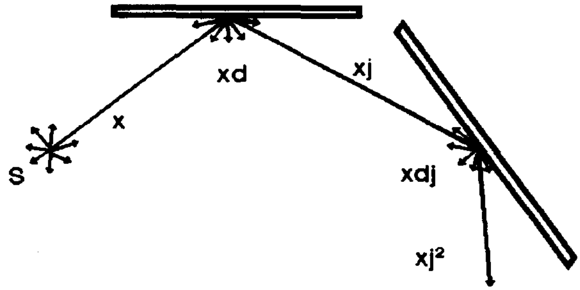

Fig. 1. Reflection and diffusion of sound by plane surfaces. j and d are the reflection and diffusion factors respectively.

which is the fraction of energy scattered in non-specular directions. In other words we define the mean geometrical energy reflection factor (J), which is defined as the ratio of the sound energy geometrically reflected from the walls to the incident one, and the mean diffuse sound energy factor (d), which is defined as the diffuse sound energy to the incident one (see Fig. 1), i.e.

$$\begin{aligned} j &= g(1-a) \\ d &= (1-g)(1-a) \\ j+d &= r \end{aligned} \tag{1}$$

where a, r, g are the absorption, reflection and geometrically reflection coefficients, respectively.

3 NOISE FIELD CALCULATION

Suppose a small room contains a white noise source lying near the centre of symmetry of the room, and an observation point inside the room. A sampling of noise with sampling period t_s equal to the time spacing of the emitted noise impulses is performed. Each noise measurement at the observation point is assumed to be the superposition of the three types of noise: the diffuse, the direct and the geometrically reflected. Consequently for each measurement at time t we can write:

$$y_t = x_t + R_t + D_t + n_t \tag{2}$$

where y_t is the total energy density corresponding to time t, x_t is the direct noise energy density at time t, R_t is the reflected noise energy density at time t, D_t is the diffuse noise energy density at time t, and n_t is uncorelated random noise.

Analytically for the received noise we can make the following assumptions.

3.1 Direct noise

For the direct sound field we can assume that the distance-receiver is small compared with the quantity Ct_s, where C is the speed of sound and t_s the sampling interval; then the air absorption, the time delay and the spherical divergence are negligible. This means that the direct sound depends only upon the emission time t and the distance between source and receiver that is assumed to be constant. In addition according to the assumption stated above this distance is small so that no delay of the received direct noise exists.

3.2 Reflected noise

In order to compute the geometrically reflected noise in a given enclosure we can use the image source principle. The general restriction for this method is that the typical dimensions of the room must be large compared to the wavelength of the highest frequency under consideration and that the walls must be plane surfaces, (though non-plane surfaces can be easily approximated by plane surfaces with sufficient accuracy). The mirror image source method is based on the construction of the mirror image of a point sound source. For image sources we can attach the following property: each image source is a directional point source. The emitted noise energy to the direction of image source-receiver is proportional to the reflected noise energy, but the emitted noise energy to all other directions is proportional to the diffuse one. When we say all other directions we mean emission to a hemisphere directed toward the observation point. With this assumption the sound field is reduced to the free field case that can be solved deterministically but indirectly.

For the number of visible image sources up to the order i we can use the approximate formula:

$$N_{\mathrm{ref}} \approx \frac{4\pi(Ct_{\mathrm{m}})^3}{3V} \tag{3}$$

where Ct_{m} is the distance of the farther image source under consideration. This equation was derived for rectangular shapes only, but it has been shown that it may be applied as well to enclosures of any shape.[1]

Following the rules of geometric acoustics we can construct the whole set of image sources (first or higher order) in the virtual space. Having the

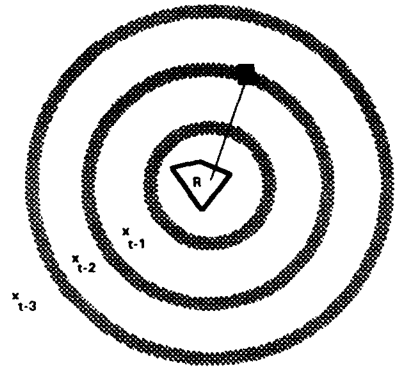

Fig. 2. Image source contribution on measured noise at time t.

observation point as centre, (see Fig. 2), we can draw spherical shells of radius r and thickness dr:

$$r_i = iCt_s, \quad \mathrm{d}r = C\mathrm{d}t, \quad i = 1, 2, \ldots$$

From the whole set of image sources, only the sources placed inside the spherical shells of radius iCt_s and thickness dr (which is supposed to be very small compared with Ct_s) contribute to the measurements taken at the time instants $t - it_s$.

According to eqn (3) it is reasonable to assume a constant volume density of image sources. For the contribution to the total energy density from image sources located inside a spherical shell, of radius r, we assume the inverse square law, so we can write:[2]

$$R_r = \frac{w_{t-i}}{4\pi C}\,\frac{\mathrm{d}N_{\mathrm{ref}}}{r^2}\,\mathrm{e}^{-mr} j^{r/l_m}\, 4\pi r^2 \mathrm{d}r = \frac{1}{V}\, e^{-mr} j_r w_{t-i}\,\mathrm{d}t \qquad (4)$$

where w_{t-i} is the source power corresponding to time $t - it_s$, R is the contribution to the geometrically reflected noise from the image sources emissions which correspond to the i shell, e^{-mr} is the air absorption, l_m is the mean free path.

Inside the virtual space only the imaginary sources up to order p located inside the successive spherical shells of radius iCt_s contribute to the geometrically observed noise at the observation point at the time instants $t - it_s$. The strength of each image source inside the spherical shell of radius Cit_s is proportional to the real source strength at time $t - it_s$. Assuming a constant sampling time t_s if we replace the time delay it_s simply by i, for the reflected noise energy density as is observed at the successive instants of time, from all spherical shells up to order p we can write:

$$\begin{aligned} R_t &= J_1 A x_{t-1} + J_2 A^2 x_{t-2} + \dots J_p A^p x_{t-p} \\ R_{t-1} &= J_1 A x_{t-2} + J_2 A^2 x_{t-3} + \dots J_p A^p x_{t-p-1} \\ R_{t-m} &= J_1 A x_{t-m-1} + J_2 A^2 x_{t-m-2} + \dots J_p A^p x_{t-m-p} \end{aligned} \quad (5)$$

where

$$J_i = j_i^{\,r/l_m}, \quad \frac{w_{t-i}}{V}\,\mathrm{d}t = x_{t-i}$$

and $A = \mathrm{e}^{-mct_s}$ is the air absorption.

In the case of symmetric absorption we can write:

$$j_i = j,\ j_2 = j^2,\ \dots j_n = j^n$$

3.3 Diffuse noise

The perfectly diffuse field is a field satisfying the criterion that the energy per unit volume and per unit solid angle of propagation direction $D(e)$ be independent of the direction e. Here we assume that the diffusion of noise builds up the diffuse noise field, or in other words the diffuse noise is that noise which is not coming directly either from the real or the imaginary sources. This assumption defines a field close to the perfectly diffuse one. During the reverberation process two conflicting mechanisms determine the magnitude of diffuse noise at time t. The first is related to the decay of noise. Assuming that the diffuse noise is uniform throughout the room, then it is decaying with time following the exponential law:

$$D_t = BAD_{t-1} \quad (6)$$

where

$$B = e^{-13{\cdot}82 t_s/T} \quad (7)$$

The index $t-1$ has the same meaning as in eqn (4).

The second mechanism is related to the reflections of noise energy. During successive reflections a flow of diffuse noise energy is added continuously into the room (see Fig. 1). For the total diffuse noise energy density at time t, as is shown in the Appendix, we can write:

$$D_t = BAD_{t-1} + FA(R_{t-1} + x_{t-1}) \quad (8)$$

where F is a function of reflection and diffusion factors which are constants for a given room. Since under the assumptions stated above the diffuse noise energy at time t is the sum remaining after the decay (eqn 8) and that created by the reflections between the time t and Δt, for the diffuse noise energy at the successively instants of time, according to the energy balance principle we can write:

$$\begin{aligned} D_t &= BAD_{t-1} + FA(R_{t-1} + x_{t-1}) \\ D_{t-1} &= BAD_{t-2} + FA(R_{t-2} + x_{t-2}) \\ D_{t-m} &= BAD_{t-m-1} + FA(R_{t-m-1} + x_{t-m-1}) \end{aligned} \tag{9}$$

Eliminating D_{t-i} by successive substitutions we can write for eqn (9):

$$\begin{aligned} D_t =& FA(R_{t-1} + x_{t-1}) + FBA^2(R_{t-2} + x_{t-2}) \\ &+ FB^2A^3(R_{t-3} + x_{t-3}) + \dots \end{aligned} \tag{10}$$

4 DIFFUSION FACTOR CALCULATION

Taking into account eqns (5) and (10) in eqn (2), we can write for y_t:

$$\begin{aligned} y_t =& x_t + (J+F)Ax_{t-1} + (J^2 + JF + BF)A^2x_{t-2} \\ &+ (J^3 + J^2F + JFB + FB^2)A^3x_{t-3} + \dots + n_t \end{aligned} \tag{11}$$

or subtracting the mean:

$$\begin{aligned} z_t =& x_t + (J+F)Ax_{t-1} + (J^2 + JF + BF)A^2x_{t-2} \\ &+ (J^3 + J^2F + JFB + FB^2)A^3x_{t-3} + \dots + n_t - [\mu + (J+F)\,A\mu \dots] \end{aligned} \tag{12}$$

The physical meaning of eqns (11) and (12) is that the received noise at the observation point inside a large enclosure is the output of a linear time invariant system with input train of equispaced in time random impulses, independent and normally distributed. The output data are not now independent because of the reverberation process. Figure 3 shows the probability density functions of emitted and received noise signals. In this figure we can see that the deviations from the zero mean of the received noise samples at the observation point is considerably smaller compared with the emitted noise samples. This is a consequence of the existence of the diffuse noise energy density. The problem can be solved statistically with the use of time series models. In these models the observations are assumed to be the output of a black box system, in which the unobservable input is a realization from a zero mean, white noise process. A moving average model $MA(p)$ is closer to the physical meaning of the reverberation process. A moving average model of order p [$MA(p)$] is a time series model in which the current deviation of a series from its mean

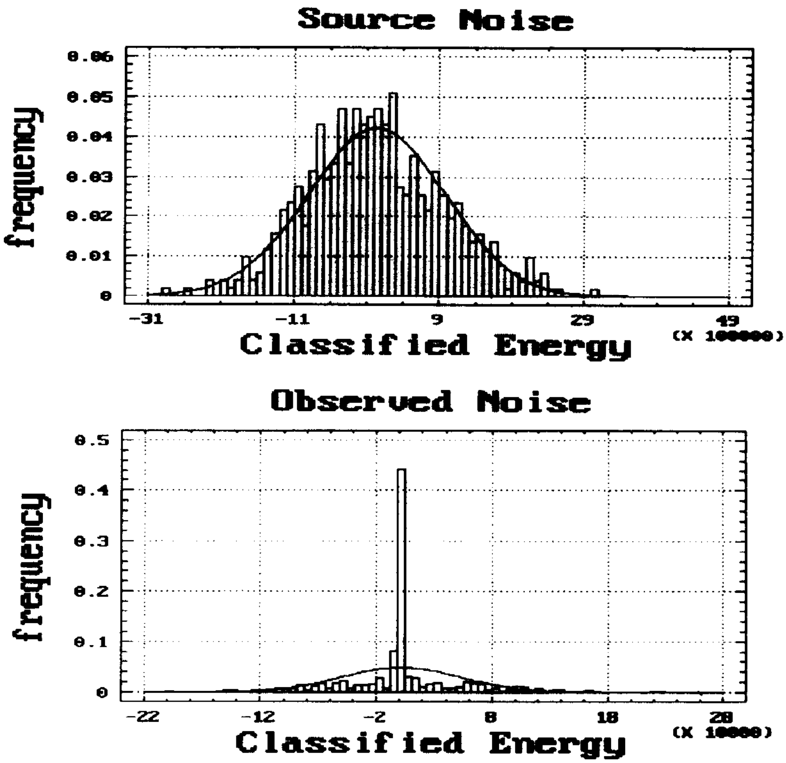

Fig. 3. Probability density functions, of emitted and received noise signal.

is a linear combination of the current and previous shocks which entered to the system. The general form of this model is:[3–5]

$$z_t = x_t - \Theta_1 x_{t-1} - \Theta_2 x_{t-2} - \Theta_3 x_{t-3} \ldots - \Theta_p x_{t-p} + y_t \tag{13}$$

Equation (13) represent a $MA(p)$ model. The use of MA models have two disadvantages: the first is that it is not possible always to obtain explicit solution for Θ_i, and the second that in the case of rooms with low absorption the order of the model (which is a finite memory model) is large, and this means high cost in computation time and low accuracy of calculations.

In order to avoid these problems we can use the autoregressive models. According to the duality property of the MA and AR processes, if the invertibility condition is fulfilled, it can be shown that the $MA(p)$ can be represented by an order q autoregressive $AR(q)$ model. An $AR(q)$ model is an infinite memory model in which the current value of the time series can be expressed as a linear aggregate of its previous values, and a random shock. The general form of an $AR(q)$ model is:

$$z_t = x_t + \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \Phi_3 y_{t-3} + \ldots \Phi_q y_{t-q} + y_t \tag{14}$$

To guarantee the invertibility of a MA(q) the solutions of the equation

$$1 - \Theta_t z - \Theta_2 z \ldots - \Theta_q z^q = 0 \tag{15}$$

have to be outside the complex unit circle.[6]

By successive substitutions of y_i in eqn (14) we finally find the *MA* inverted form of the *AR* process:

$$z_t = x_t + \Phi_1 x_{t-1} + \left(\Phi_1^2 + \Phi_2\right) x_{t-2} + \left(\Phi_1^3 + 2\Phi_2\Phi_1 + \Phi_3\right) x_{t-3} + \ldots n'_t \tag{16}$$

Comparing eqns (12), (14) and (16) we can write:

$$\begin{aligned} -\Theta_1 &= \Phi_1 = (J + F)A \\ -\Theta_2 &= \Phi_1^2 + \Phi_2 = (J^2 + JF + BF)A^2 \end{aligned} \tag{17}$$

Figure 4 shows the whole process in terms of energy flow in the time domain for a room, with symmetric absorption. In this figure x_{t-1}, $x_{t-2}, \ldots$ are the emitted noise impulses from imaginary noise sources corresponding to times $t - \Delta t$, $t - 2\Delta t, \ldots x_t$ is the real source emission at time t. The upper row corresponds to the diffuse energy flow, and the lower row corresponds to the flow of speculary reflected noise energy. The blocks in the middle feed the room with new diffused noise energy, due to successive reflections. The flow of energy corresponds to a narrow band noise signal.

In constructing the $AR(q)$ model we can use the partial autocorrelation (pacf) to tell us how many parameters to include in the model. If the cut

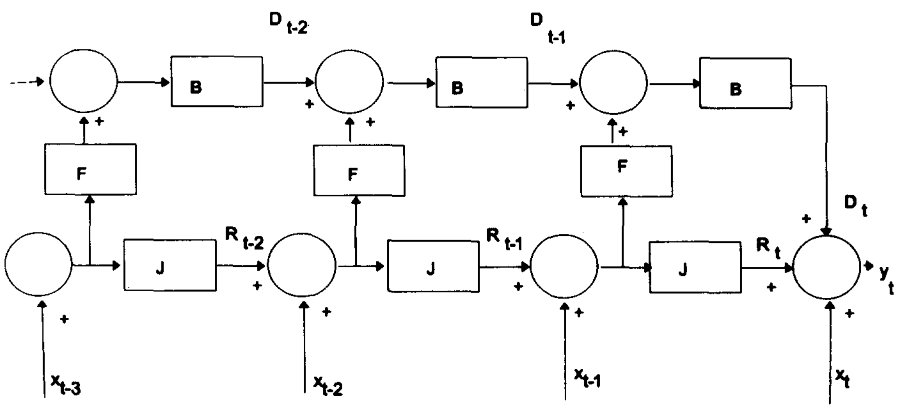

Fig. 4. Energy flow diagram from real and imaginary noise sources in time domain. J, F, B are dimensionless constant factors.

off lag of the sample pacf is p there is a strong evidence that an $AR(q)$ model in appropriate. For a more precise determination of the model order here we used the Akaike's final prediction error (FPE).[7] The FPE computes a measure over all possible model orders and chooses the model order that minimizes the measure. For an AR process the FPE is defined as:

$$FPE_q = \frac{n + (q+1)}{n - (q+1)} s_q^2 \tag{18}$$

where s_q^2 is the residual sum of squares. Using this criterion, an estimate p is chosen such that

$$FPE(p) = \min\{FPE(k) | k = 1, 2, \ldots m\}$$

Summing all stated above, the solution of the following non-linear equation will give us the value of the diffusion factor:

$$\Phi_1 = Aj^N + Ad \frac{j^N - b^N}{j - b} \tag{19}$$

Where N is the mean number of successive reflections during the sampling interval, given by:

$$N = \frac{CSt_s}{4V} \tag{20}$$

Equation (19) can be solved numerically. The required parameters for the evaluation of the degree of the diffusion are: the room's dimensions, the reverberation time (T_{60}), the reflection or absorption coefficient, the sampling interval, and the autoregressive coefficient of x_{t-1}, Φ_1.

In some cases the value of absorption coefficient is unknown. The decay curve can be used for an approximate evaluation of this value. The decay curve in logarithmic scaling is not always a straight line. In most cases we can distinguish more than one slope (see Fig. 5).[8–12] Since the geometrical reflected noise has short duration, compared with the reverberation time, we can assume that generally the first, and possibly the second or third, slope corresponds to the decay of the total noise field, and the last distinct slope to the pure diffuse one. An extrapolation of the first and last slope evaluates two different reverberation times (T_{60}), an early and a late one. The early reverberation decay time can be used for the calculation of the mean absorption coefficient using, for example, the Eyring's or Sabine's

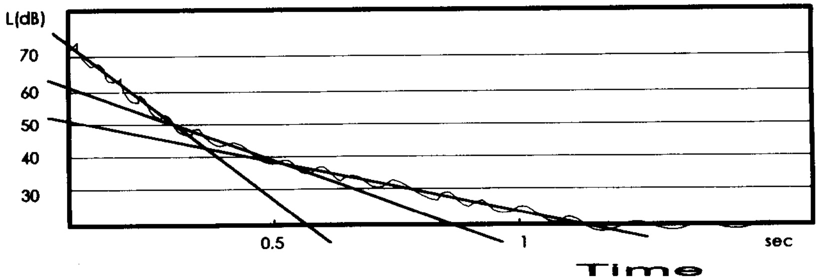

Fig. 5. Decay curve of noise in a room.

formula. The late one can be used for the diffuse noise energy decay. Eyring's formula which relates the mean absorption coefficient a to the reverberation time when the air absorption is taken into account is given by:[13]

$$T = \frac{0{\cdot}161V}{4mV - S\ln(1 - a)} \tag{21}$$

Numerical example

To confirm experimentally the validity of the method an experiment was carried out in an empty small amphitheatre of the University of Patras, in the absence of background noise and flutter echo. The room's volume was 500 m^3, with total wall area of 460 m^2. The excitation signal (a series of equispaced in time, normally distributed impulses) was written on a tape recorder. As noise source an omnidirectional loudspeaker was used. A second tape recorder was used to pick up the received and emitted noise signals simultaneously. An omnidirectional microphone was used as receiver placed in different points near the centre of symmetry of the room under test. Figure 6 shows the experimental set-up. An extrapolation of the measured Early Decay Time (EDT) which is based on an evaluation of

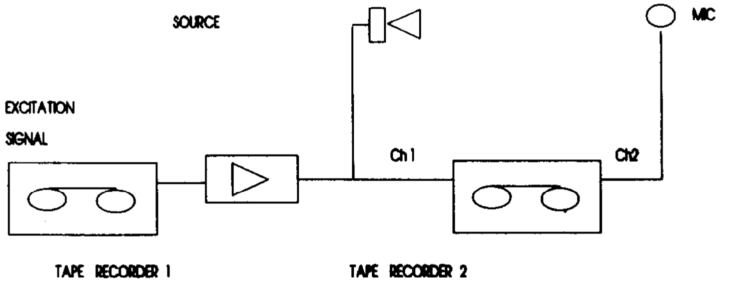

Fig. 6. Experimental set-up.

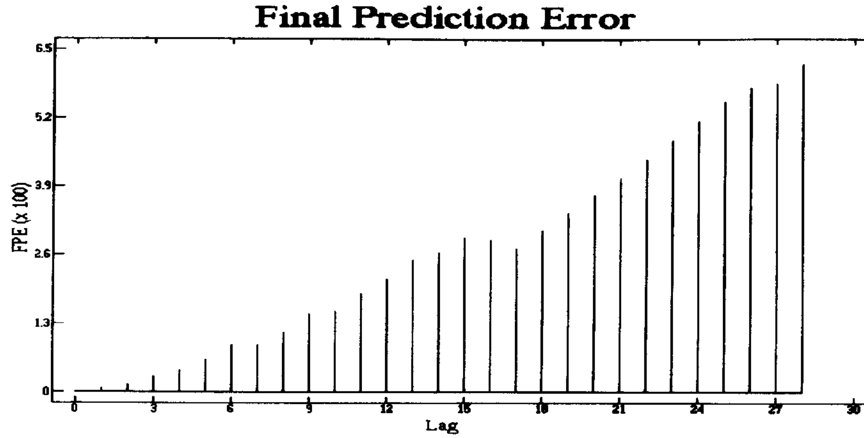

Fig. 7. Application of *FPE* for model selection.

the decay in the range from 0 to −10 dB, gave a reverberation time T_{60} = 1·45 sec, and a late one at 1·55 sec. Under the experimental conditions (temperature 25°C, centre frequency f_c = 2000 Hz) the corresponding air absorption factor was $4m$ = 0·0114 m^{-1}. Application of the Eyring formula indicates a mean total absorption coefficient of 0·1. Figure 3 shows the emitted and received noise signal consisted of 512 noise samples for the above room and sampling interval t_s = 0·1 sec. The *FPE* evaluation for model selection and the calculation of the autoregressive coefficients, was made by a computer program, based on the algorithm described in Ref. 7. Figure 8 shows the *FPE* for the room under test which suggests an $AR(1)$ model. The solution of eqn (18) gives Φ_1 = 0·46. A numerical solution of eqn (19) gives d = 0·47.

5 DISCUSSION

In this paper we try to find statistically a relation between the diffuse and the reflected noise by reflecting surfaces in order to evaluate the degree of the diffusion of incident rays on reflecting surfaces. The performance of each room can be described by a $MA(\infty)$ model or according to the invertibility property of time series by an $AR(p)$ model. The order of the model depends upon the reverberation time T_∞ and sampling interval t_s. In most cases use of *FPE*, which gives the order p, gave the values p = 1 but in some cases p was 4–10. Here we assume that the change of slope in the reverberation decay curve, if there is no flutter echo, is caused by the elimination of the geometrically reflected noise which has very short life compared with the perfectly diffuse noise. Since in most cases the observed slopes has no definite limits, an extrapolation of the EDT and the last 10 dB decay (−50 to −60) dB can be used. The numerical solution of eqn (19) in some cases can give more than one acceptable real root. In this case a repetition of the experiment, with smaller sampling interval reduces the degree of eqn (19). The great numbers of objects that are located inside the room and the shape of the reflecting walls are the main reasons for the deviations of calculated diffuse noise from the real one. The results presented above are for a particular room. They can be applied to any other rooms as well. Although the excitation signal was a narrow band noise signal centred at a frequency of 2 kHz, the dependence on the frequency of the calculated parameters was not examined. The application of the method to rooms with parallelepiped shape, which indicates a higher order of autoregressive models, needs further investigation.

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APPENDIX

Assume a room of volume V and total surface area S. At time t the source emits a random shock. For large values of sampling interval t_s the number of reflections N during this interval is nearly equal to

$$N = \frac{t_s}{t_c} = \frac{Ct_s}{l_c} \tag{A1}$$

where t_c is the mean free path time and l_c is the mean free path given by:[12]

$$l_c = \frac{4V}{S} \tag{A2}$$

Suppose that the energy density corresponding to direct and geometrically reflected noise at time t is D_{Dt}. Then:

$$D_{Dt},\ je^{-mct_c}D_{Dt},\ j^2e^{-2mct_c}D_{Dt},\ j^3e^{-3mct_c}D_{Dt},\ \ldots,\ j^Ne^{-Nmct_c}D_{Dt} \tag{A3}$$

are the reflected and not absorbed (by the air) energy density.

This process supplies new diffuse noise energy to the room. After each reflection the diffuse noise energy density is increased by the quantity.

$$dD_{Dt},\ dje^{-mct_c}D_{Dt},\ dj^2e^{-2mct_c}D_{Dt},\ dj^3e^{-3mct_c}D_{Dt},\ \ldots,$$
$$dj^Ne^{-Nmct_c}D_{Dt}$$

Supposing that each of this quantity is decaying independently with time following the exponential law, according to the energy balance equation, the totally remaining diffuse noise energy at the end of the considered interval (t_s) is:

$$\begin{aligned} &b^{N-1}de^{-mct_c}D_{Dt} + b^{N-2}de^{-2mct_c}jD_{D(t-1)} \\ &+ b^{N-3}de^{-3mct_c}j^2D_{D(t-1)} + \ldots + j^{N-1}de^{-(N-1)mct_cD_{D(t-1)}} \end{aligned} \tag{A4}$$

where $b = e^{-13{\cdot}82t_c/T}$, which implies: $B = b^N = e^{-13{\cdot}82t_s/T}$.

Taking into account the air absorption we can write for (A4):

$$\begin{aligned} &b^{N-1}de^{-mct_c}e^{-(N-1)mct_c}D_{Dt} + b^{N-2}de^{-2mct_c}e^{-(N-2)mct_c}j\,D_{D(t-1)} \\ &+ b^{N-3}de^{-3mct_c}e^{-(N-3)mct_c}j^2\,D_{D(t-1)} + \ldots + j^{N-1}de^{-mct_c}D_{D(t-1)} \end{aligned} \tag{A5}$$

After some algebraic calculations we can write for (A5):

$$\begin{aligned}\mathrm{d}D_{Dt}\mathrm{e}^{-(N-1)mct_s}[j^{N-1}+bj^{N-2}+b^2j^{N-3}+\ldots b^{N-1}] \\ = \mathrm{d}D_{Dt}\mathrm{e}^{-Nmct_s}\frac{(j^N-b^N)}{j-b}\end{aligned} \tag{A6}$$

Thus the diffuse noise energy can be written:

$$D_t = BAD_{t-1} + FA(R_{t-1} + x_{t-1}) \tag{A7}$$

where

$$F = \mathrm{d}\frac{[j^N - b^N]}{[j-b]} = \mathrm{d}\frac{J-B}{j-b} \tag{A8}$$